Dynamic Aggregation of Heterogeneous Interacting Agents and Network: An Analytical Solution for Agent Based Models

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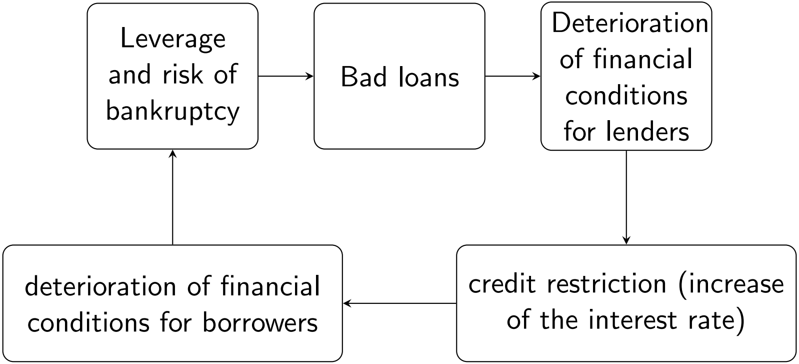
- Economics' methodology inspired by Newton-Euler Classical Mechanics
- Poinsot *Elements of Statics* (1803) $\mapsto$ Walras *Elements* (1874)
  - reductionism, determinism and mechanicism
  - micro = MACRO
- Incomplete information, heterogeneity, interaction $\mapsto$ money and time
- Microfoundations: taking Lucas seriously
  - **RA** models: Hartley (1997); Kirman (1992)
  - **ABM**: computation: Judd and Testfasion (2008)
  - **ME**: stochastic aggregation and analytic solution of HIA
- With the RA framework one cannot handle financial fragility and robustness because RA rules out the coordination problem
- **Beyond simulations with ME**: Aggregation of HIA whose dynamics can be analytically modelled
- We propose a mathematical tool for modelling HIA models: the finite Markov chain solved analitycally by a ME
- **Emergence**: the spontaneous formation of self-organized structures at different layers of a hierarchical system configuration.
- The equilibrium of a system no longer requires that every single element be in equilibrium by itself, but rather that the statistical distributions describing aggregate phenomena be stable.

- Financial fragility (Minsky (1982); Greenwald and Stiglitz (1988, 2003))



"The high rate of bankruptcy is a cause of the high interest rate as much as a consequence of it" (*Towards a new paradigm in monetary economics*: 2003, 143)

- Financially constrained production function:

$$Q_{f,t} = \alpha A_{f,t}^{\beta} \tag{1}$$

where $\alpha > 0$ and $\beta \in (0, 1)$ and A is net worth.

- Consequently, the demand for labour is

$$N_{f,t} = \gamma Q_{f,t} \tag{2}$$

- The wage bill W is

$$W_{f,t} = wN_{f,t} \tag{3}$$

the nominal wage w is a constant, N_f is the quantity of work.

- Because of asymmetric information there is hierarchy in the source of financing: first internal finance, then debt (equity rationing)
- We can identify 2 types of firms:
  - Self Financing firms (SF): $A_{f,t-1} \geq W_{f,t}$
  - Non Self Financing firms (NSF): $A_{f,t-1} < W_{f,t}$.
- Only NSF firms are nodes of the network;
- The demand for credit for NSF firms is

$$D_{f,t} = W_{f,t} - A_{f,t-1}. \tag{4}$$

- The selling price is a stochastic realization of a uniform distribution (Greenwald and Stiglitz, QJE 1993)

$$p_{f,t} \rightarrow U(u_{min}; u_{max}). \tag{5}$$

- Firms profits $\pi_{f,t}$ are defined as

$$\pi_{f,t} = p_{f,t} Q_{f,t-1} - W_{f,t} - r_{f,t} D_{f,t}. \tag{6}$$

- The net worth of firms is the sum of past profits:

$$A_{f,t} = A_{f,t-1} + \pi_{f,t} \tag{7}$$

- If $A < 0$ the firm goes bankrupt and it is replaced by a new one.

1. The firm demands for credit to the bank which offers the **lowest interest rate**.
2. The bank provide credit to the firm, establishing a **link** (no credit limit for the bank).

- The interest rate asked by bank b to firm f

$$r^f_{b,t} = r_{f,t} = a\left[\left(\frac{D^s_{b,t}}{A_{b,t}}\right)^a + \left(\frac{D_{f,t}}{A_{f,t}}\right)^a\right] \; ; \quad a > 0 \tag{8}$$

where $D^s_{b,t}$ is the total lending, $A_{b,t}$ is the bank's net worth.

- Loans supply is modelled as the Basel 2 criteria, according to

$$D_{j,t} = kA_{j,t} \tag{9}$$

where k is a multiplier set by the monetary authorities.

- The profits of the bank b are given by

$$\pi_{b,t} = \sum_{f \in F^{B,b}_t} r_{b,f,t} D_{b,f,t} - E_{b,t} r^E \tag{10}$$

- while banks net worth is computed as

$$A_{b,t} = A_{b,t-1} + \pi_{b,t} - BD_{b,t} \tag{11}$$

where BD is the bad debt.

- If $A_{b,t} < 0$: the bank goes bankrupt, and it is replaced by a new one.

The analytical solution of the model is articulated in three steps:

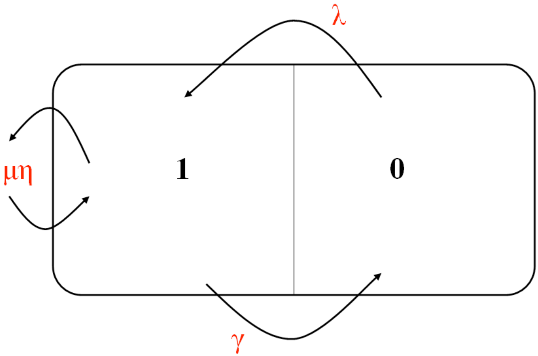
1. the quantification of the number of NSF firms by means of the first ME;
2. the identification of the degree distribution and the study of the dynamics of the average degree solving a second ME;
3. the second ME feeds back into the first one (the system is a nested MEs one).

The method: stochastic dynamic aggregation

How to aggregate *heterogeneous* and *evolving* agents?

1. Agents are classified into different **micro-states**, according to their characteristics;
2. A *representative* agent for each cluster is identified (**Mean-field interaction**);
3. Macro configuration is identified by the *number of agents that occupy each micro-state at a given time* (the **macro-state**), governed by a *stochastic law* (jump Markov process);
4. This stochastic law is functionally modelled as a **master equation** (ME).

How many NSF firms?



- State 0: SF;
  State 1: NSF;
- γ: transition rate from 1 to 0; λ: transition rate from 0 to 1;
- μ: probability of bankruptcy;
- η: probability of being in 1;
- Hp: $\mu \equiv \eta$.

Transition rates: probability to get, in a given unit of time, a "jump" of an agent from one state to another:

$$\boxed{\begin{aligned} \lambda &= \zeta\eta \\ \gamma &= \iota(1-\eta) \end{aligned}} \tag{12}$$

where:

- ζ and ι: **transition probabilities** for a firm to move from NSF to SF and from SF to NSF;
- η: the a-priori probability for a firm to be in state 1.

Identification of the two *mean field* firms

- In order to calculate the transition probabilities we identify a NSF and SF *mean field* firm;
- we can calculate an average net worth for each the two clusters of firms (calling these average values A_0 for SF and A_1 for NSF firms);
- accordingly, we determine the desidered output (Q_0 and Q_1) as we were dealing with only two firms rather than with N;
- we can express the transition rates as a function of the stochastic price shock $u \in [u_{min}, u_{max}]$, the rate of interest and leverage (i.e. debt and equity base).

NSF $\longrightarrow$ SF

- The probability ι depends on the capacity of the firm of having at time $t-1$ a profit large enough to pay the salary bill and the outstanding debt:

$$A_{1,t-1} + u_{1,t} Q_{1,t-1} - (1 + r_f) D_{1,t-1} \geq W_{1,t} \tag{13}$$

- if the price is above a certain threshold, the firm can obtain a profit sufficient to became SF. Indicating this threshold with $\bar{u}$, we can write

$$u_{1,t} \geq \frac{W_{1,t} + (1 + r_1, t) D_{1,t-1} - A_{1,t-1}}{Q_{1,t-1}} = \bar{u}_t. \tag{14}$$

- As the price u is an exogenous stochastic quantity with a known distribution, it is possible to express condition (13) in terms of the probability function for $u_{f,t}$ as

$$\iota_t = F(\bar{u}) = \frac{\bar{u} - u_{min}}{u_{max} - u_{min}}. \tag{15}$$

SF $\longrightarrow$ NSF

- Analogously, the condition for the representative SF firm to become NSF can be written as:

$$A_{0,t} + u_{0,t} Q_{0,t-1} < W_{0,t}. \tag{16}$$

- For this transition to occur, the price shock should be below a level $\underline{u}$:

$$u_{0,t} < \frac{W_{0,t} - A_{0,t-1}}{Q_{0,t-1}} = \underline{u}. \tag{17}$$

- the probability ζ can be written as

$$\zeta_t = 1 - F(\underline{u}) = 1 - \frac{\underline{u} - u_{min}}{u_{max} - u_{min}}. \tag{18}$$

The macro-dynamics

ME: quantifies the variation of the probability of observing N_1 agents in state 1 in a small interval of time:

$\frac{dP(N_1,t)}{dt}$ =(inflows of probability fluxes into state 1)-(outflows of probability fluxes out of state 1)

$$\frac{dP(N_1,t)}{dt} = \underbrace{\lambda[N-(N_1-1)]P(N_1-1)+\gamma(N_1+1)P(N_1+1)}_{\text{inflows of probability}} - \underbrace{[(\lambda+\gamma)N]\,P(N_1)}_{\text{outflows of probability}} \tag{19}$$

Evaluating the components of the dynamics

- Since a solution of the ME is possible only in very particular conditions, following Aoki (2002) we split the state variable N_1 in two components:
  - the **drift** (m): trend value for $n_1 = N_1/N$;
  - the **spread** (s): aggregate fluctuations around the trend;
  - for the sake of simplicity,we assume:

$$N_1 := Nm + \sqrt{N}s \tag{20}$$

- In this way we are able to identify the dynamics of the trend and the probability of the fluctuations around this trend.

Asymptotic solution

- *Macroscopic equation* (the drift):

$$\boxed{\frac{dm}{dt} = \lambda m - (\lambda + \gamma)m^2 \quad \Rightarrow \quad m^* = \frac{\lambda}{\lambda+\gamma}} \tag{21}$$

- *Fokker-Planck equation* (the spread), whose solution yields the probability density of fluctuations:

$$\boxed{p(s) = C \exp\left(-\frac{s^2}{2\sigma^2}\right)} \tag{22}$$

where $\sigma^2 = m^* \frac{\gamma}{\lambda+\gamma}$

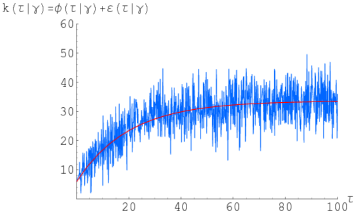


Figure: Solution of the ME: dynamics of the drift with fluctuations distributed according to the FP equation.

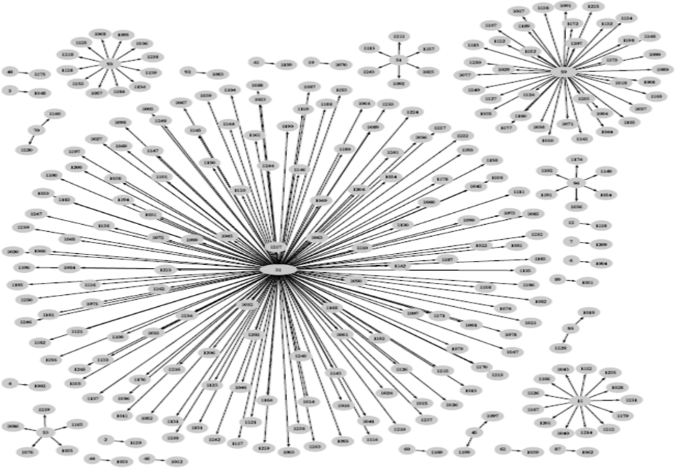


Figure: Network of islands (cliques) around banks. Degree distribution tends to a power-law

Analytical solution of the model shows that:

- the model generates endogenous cycles and fluctuations, domino effects and financial crises, with avalanches of firms' and banks' failures;
- a higher concentration in the banking sector (degree) makes the system more fragile;
- there is a non-linear relation between the banks degree and output growth;
- positive correlation between the variances of the degree and of the output: changes in the network structure, due to failures imply a higher volatility of output (TITF); i.e. a higher concentration brings about a larger volatility.
- positive correlation between the giant component (biggest bank) and size of bankrupted banks: the presence of a big bank can destabilise the system (TBTF);
- All the systemic risk measures we need (ECB): (i) degree of connectivity; (ii) degree of concentration; (iii) size of exposures.

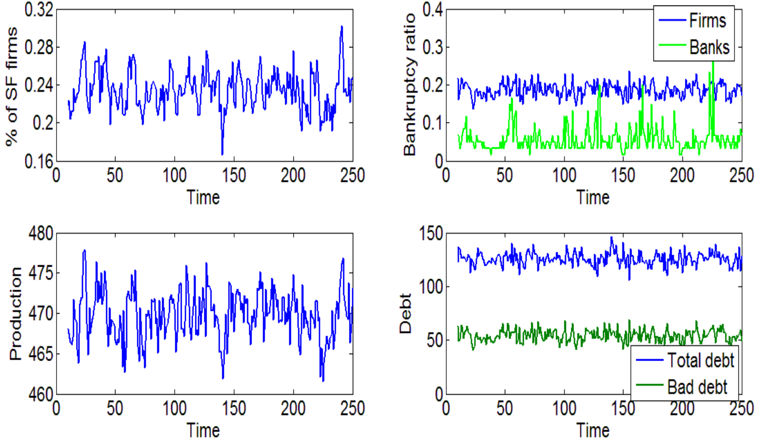


Figure: Share of self financing firms; firms' and banks', bankruptcy ratios; and aggregate production.

- In this paper we take one step toward building approximate models of HIA.
- The idea is to neither ignore interactions between agents, nor to get hopelessly mired in complicated details by trying to model those interactions in their completeness, but to strike for some middle ground in which the consequences of interconnectedness can be at least crudely assessed.
- To achieve this, we start with a model of the economy as a credit network, in which firms interact directly with banks.
- Rather than simulating these interactions, we describe them with probabilities, and derive equations describing the evolution of the network, how its structure changes with time.
- The equations we derive gives some qualitative insight into the systemic fragility of the credit network.

- We find that rising economic output (reflecting in increase in the overall wealth of the firms) turns out to be proportional to how much the loans in the system come to be concentrated among a few banks. In network terms, this concentration can be measured as the average degree for banks in the network.
- There's a natural tendency for this number to rise as the economy expands and banks and firms profit.
- This rise in concentration also comes with a cost: the chance failure of one bank brings trouble to many firms, which pass it on to other banks, leading to further failures.
- Cascades of failures put financial pressure on all firms, raising the costs of borrowing and slowing the economy.
- The feedback between economic growth, rising interconnectedness and eventual systemic collapse leads to booms and busts.

Some references

1. Gardiner (1985) Handbook of Stochastic Methods - Springer
2. Risken (1989) Fokker-Planck Equations - Springer
3. Foley (1994) A Statistical Equilibrium Theory of Markets, Journal of Economic Theory, 62(2)
4. Aoki (1996) New Approaches to Macroeconomic Modeling. Evolutionary Stochastic Dynamics, Multiple Equilibria, and Externalities as Field Effects - Cambridge
5. Aoki & Yoshikawa (2006) Reconstructing Macroeconomics. A Perspective from Statistical Physics and Combinatorial Stochastic Processes - Cambridge
6. Di Guilmi, Gallegati and Landini (forthcoming) Interactive Macroeoconomics.