Anomalous Price Impact and Critical Liquidity: Theory and ABM

Bence Tóth[a] and Jean-Philippe Bouchaud[a]

[a]Capital Fund Management, Paris, France

Paris, September 10, 2011

Collaborators from CFM

- Yves Lemperière
- Cyril Deremble
- Joachim de Lataillade
- Julien Kockelkoren

Outline

Price impact

What is price impact?

- Price impact: Correlation between the sign of an arriving order and the subsequent price change.
- In general, buyer (seller) initiated trades push the price up (down).
- Two stories: trades are informed and predict price changes or even random trades (statistically) impact prices?

Several types of price impact

- Single trade impact vs. impact of a metaorder.
- Impact of the average order flow
- Immediate impact vs. longer time impact (impact decays)

Metaorders

A metaorder is a **series** of connected trades in the same direction from the same initiator.
Reason: when trading a large position, typically we have to **split** it up and trade incrementally.

- Not enough liquidity available at any given moment: for a liquid stock the instantaneous volume in the order book is $\approx 10^{-5}$ of market capitalisation, while the total volume traded in a day is $\approx 5\,10^{-3}$
- Evidence for order splitting can be found from trade sign correlations and sometimes from traders ID

Impact of metaorders

Widely observed result [Almgren, Engle, JPM, DB, Lehmann, CFM]

A metaorder of size Q has a price impact:

$$I(Q) = Y\sigma_T \left[\frac{Q}{V_T}\right]^{\delta}, \tag{1}$$

where

Q is the volume of the metaorder

σ_T is the volatility of the market

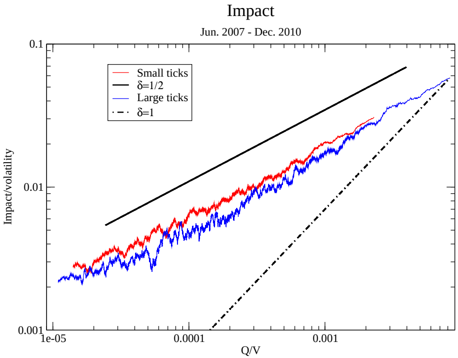
V_T is the total volume traded in the market

$\delta \in [0.4, 0.7]$; Y is a constant of order unity

Remarkable stability of results (style of trading, strategies, markets, periods, tick sizes, treatment of data..)

$$I(Q) = Y\sigma_T\sqrt{\frac{Q}{V_T}}$$

Results from CFM trades:



Note: Singular for $Q \to 0$! Trading $\sim 1\%$ of daily volume moves the price by $\sim 10\%$ of daily volatility!

$$I(Q) = Y\sigma_T\sqrt{\frac{Q}{V_T}}$$

Importance of the square-root law

- Microscopic ingredient for price formation & tâtonnement
- An important item for building ABM of financial market: price impact
  Note: classic models are linear! (Beja-Goldman, Kyle, Lux-Marchesi, Giardina & JPB, etc.)
- Crucial for controling cost of trading
- Concave is not additive ($I(Q_1 + Q_2) \neq I(Q_1) + I(Q_2)$): but $Q \ll V$, how can the response be non-linear?
- Diverging susceptibility for small volumes: small orders can generate large responses

What is the intuition behind the concavity?

After having traded $Q/2$, the next $Q/2$ will have less impact
$\Rightarrow$ this means that there is increasing volume available deeper in the book
However the typical order book that one **observes** is not like that.

There has to be some **latent** volume that only appears when we push the price.

Suppose the latent volume grows linearly vith depth:
$V_{\pm}(p) \propto |p - p_0|$

$$Q = \int_{p_0}^{p_0+I} V_{+}(p)dp \longrightarrow p = p_0 + a\sqrt{Q} \qquad (2)$$

Note: $V_{\pm}(p \rightarrow p_0) \rightarrow 0 \ldots$

The idea of a latent order book

Assumptions

- Between time t and $t + dt$, when price is $p(t)$, some investors decide to put a **latent** limit order of size v at a price $p(t) \pm \Delta$ with probability $\lambda_{\pm}(\Delta, v)dt$
  $\Rightarrow$ these orders are latent, they may only appear in the real book when the price approaches $p(t) + \Delta$
- Investors can also cancel their latent orders: $\nu_{\pm}(\Delta, v)$
- The price, $p(t)$, fluctuates due to the effect of market orders

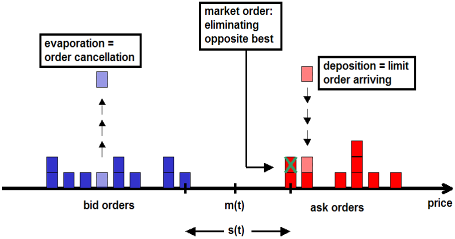
Basics of the model

Some (innocuous) simplifications

- The basic model is essentially that of E. Smith et al. (2003)[a]
- Latent limit orders arrive with a **constant, flat** rate per unit time per unit price over an infinite support $(\lambda_{\pm}(\Delta, v) \equiv \lambda)$ $\Rightarrow$ no Δ, v dependence
- Cancellation: **constant, flat** rate of limit orders being removed per unit time $(\nu(\Delta, v) \equiv \nu)$ $\Rightarrow$ defines a lifetime
- Market orders: rate μ per unit time
- Discretization: tick size and unit volumes

[a] E. Smith, J.D. Farmer, L. Gillemot, S. Krishnamurthy, Quant. Fin. 3(6) (2003)

Kinematics of the model



- The depth of the book far from the mid: $\rho_\infty = \lambda/\nu$:
  "Deep" markets: $\rho_\infty \gg 1$
- The speed of the market is defined by ν/μ:
  "Slow" markets: $\nu/\mu \ll 1$, many trades with a frozen latent book

Analytical treatment

Let $\rho(u)$ be the density of the book at a distance u away from the best price. Let's assume that the price process is **diffusive**.
In the (moving) reference frame of the mid price, $u = p - p(t)$:

$$\frac{\partial \rho}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \rho}{\partial u^2} + \lambda - \nu \rho \tag{3}$$

$$\Rightarrow \rho_{st}(u) = \rho_\infty \left(1 - e^{-u/u^*}\right); \qquad u^* = \sqrt{\frac{\sigma^2}{2\nu}} \tag{4}$$

Close to the current price ($u \to 0$): $\rho(u) \approx \rho_\infty \frac{u}{u^*}$.
Generic behaviour, valid whenever rates are regular around $u = 0$, provided prices are diffusive!

Analytical treatment

Linearity of the book

- The linear regime extends roughly up to a distance u^*, which is of the order of the price variation on the scale of the lifetime of an order – say a few hours to a few days, or on the scale of "price sensitivity", in both cases a few %
- There is a liquidity "trough" around the price: the local liquidity is $\sim (\mathrm{tick}/\mathrm{u}^*)^2$ smaller than the daily volumes (numerically: $(\mathrm{tick}/\mathrm{u}^*)^2 \sim 10^{-3}$)
- This has two consequences: splitting of metaorders and square-root impact!!
- But in the above analytic calculation, diffusivity was **assumed**!

The idea again

Latent volume

Not the volume revealed in the real order book. Only intended volumes that would reveal if the price was to come closer.

Correlated trades & price diffusion

Clearly trades have to be visible: they appear in the **real** order book.

- But in the model trade signs at this point are not correlated!
  $\Rightarrow$ we need to add autocorrelation in the direction of trades
- Increasing volume away from the best prices: the basic model is mean reverting, prices are subdiffusive.
  $\Rightarrow$ we need to obtain a diffusive price process

Further ingredients

Autocorrelation

LMF model [a]:

Generate runs of $+/-$ signs, with length distribution $P(L) \propto L^{-(\alpha+1)}$.

Can be shown to lead to a power law autocorrelation of trades, *as observed empirically*

$$C(\ell) \propto \ell^{-\gamma}, \text{ where } \gamma = \alpha - 1 \quad (5)$$

Leads to long-memory of the signs for $1 < \alpha < 2$, and provides a realistic background.

[a] F. Lillo, Sz. Mike, J.D. Farmer, Phys. Rev. E 71, 066122 (2005)

But with unit volume execution, deep and slow markets are always subdiffusive, $\forall\gamma$!

Further ingredients

Conditioning on volume

For real markets: the **fraction** of volume taken from the opposite best, is (roughly) constant.

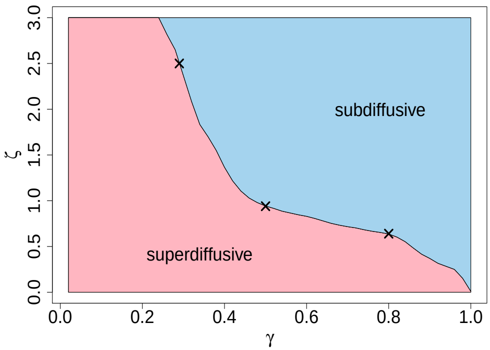
To mimic this behaviour we define the following probability of the "eaten" fraction f:

$$P(f) = \beta(1-f)^{\beta-1} \tag{6}$$

- $\beta \rightarrow 0$: all volume is taken
- $\beta = 1$: flat distribution of volume taken
- $\beta \rightarrow \infty$: minimum (unit) volume is taken

Diffusion "map"

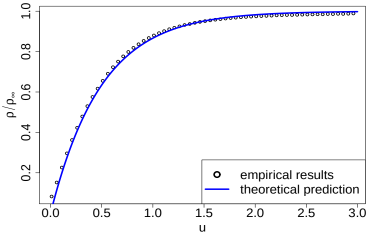
Subdiffusive vs. superdiffusive regions
"Efficient" (diffusive) boundary (what is the exact shape?)
Small γ, small β favors superdiffusion



Summary of the model

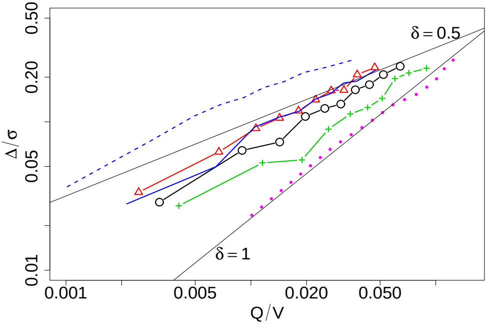
A model, that aims at describing the latent liquidity. We have:

- long-memory of trade signs
- opposing flow of limit orders
- diffusive prices
- on average linear book



Introducing metaorders

Now we have the background model, we can introduce metaorders, and study their impact (with MOs)



Introducing metaorders

For realistic parameters

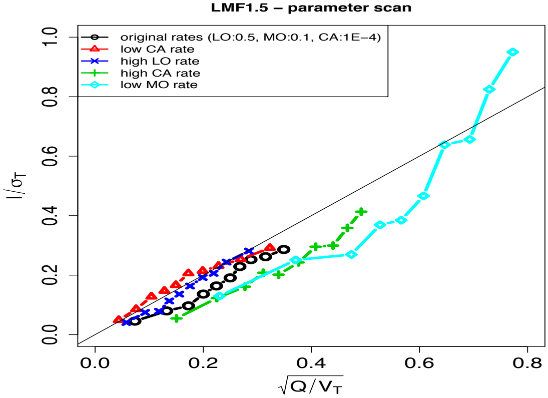
- the model reproduces a strongly concave impact

$$I(Q) = Y\sigma_T \left[\frac{Q}{V_T}\right]^{\delta}, \qquad \delta \in [0.5, 0.7]$$

- weak dependence on the autocorrelation exponent
- no dependence on participation rate (down to 2% participation rate)
- Y-ratio is of order unity (between $0.8 - 1.1$)

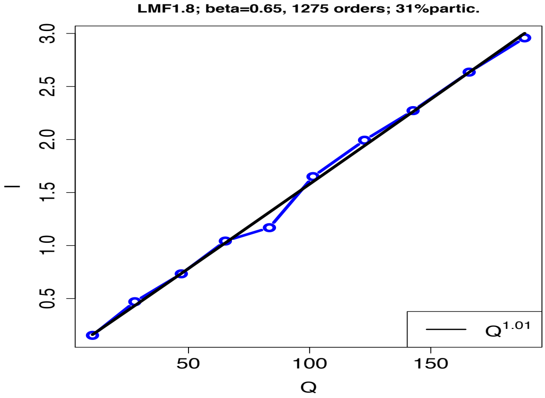
Stability of the results

Varying the parameters λ, μ, and ν.
When diffusive prices can be achieved and $\tau_{execute} \ll \tau_{life}$, we always find the same form.



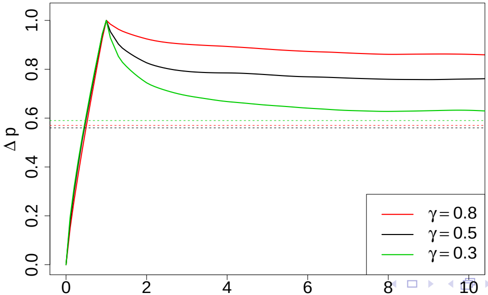
The case of fast relaxation

If $\tau_{execute} > \tau_{life}$ we get linear dependence: $I(Q) \propto Q$, as one should expect.



Decay of impact

After the finishing a metaorder, our impact decays. In event time seems to go down to approx $0.6\times$ temporary impact, similar to the empirical results of Moro et al. (2009).
Note: the plateau is above the "fair price" (average price paid during the metaorder), contrary to the tenet of a recent theory by Farmer et al.



Summary

Empirical fact:

- Impact of metaorders is a concave function of volume

Model & Results:

- Latent volume taken into account
- Very few behavioural assumptions (no fundamental price, no market makers, no adverse selection)
- Analytic theory of linear liquidity profiles
- Trade correlation + liquidity wall = Diffusive price
- Concave impact, stable against change of parameters

Conclusion

The critical nature of liquidity

- Local liquidity is vanishingly small by necessity! (it is eaten by the diffusive motion of prices)
- This imposes a splitting up of metaorders and long-range memory in the sign of trades...
- ...and leads to a breakdown of linear response and an anomalously large impact for small trades ($\Rightarrow$ concave impact)
- Liquidity fluctuations are bound to play a crucial role $\rightarrow$ microcrises and jumps in prices – without news (Joulin et al.)